

# Basic Properties of MHD turbulence in the Inertial Range

Andrey Beresnyak<sup>1,2</sup>

<sup>1</sup>*Los Alamos National Laboratory, Los Alamos, NM, 87545*

<sup>2</sup>*Ruhr-Universität Bochum, 44780 Bochum, Germany.*

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## ABSTRACT

We revisit the issue of spectral slope of MHD turbulence in the inertial range and argue that numerics favor Goldreich-Sridhar  $-5/3$  slope rather than  $-3/2$  slope. We also did precision measurements of anisotropy of MHD turbulence and determined the anisotropy constant  $C_A = 0.34$  of Alfvénic turbulence. Together with previously measured Kolmogorov constant  $C_K = 4.2$ , or 3.3 for purely Alfvénic case, it constitutes a full description of MHD cascade in terms of spectral quantities, which is of high practical value for astrophysics.

**Key words:** MHD – turbulence.

## 1 INTRODUCTION

Astrophysical plasmas are often described as ideal MHD fluid – a perfectly conducting, inviscid fluid described by MHD equations. The use of continuous fluid approach and absence of dissipation terms is motivated by the fact that astrophysical scales are typically humongous compared to the molecular and plasma mean free paths and the microscopic dissipation scales<sup>1</sup>. MHD can be successfully applied to all phases of the interstellar medium (ISM), large scales of molecular clouds, intracluster medium (ICM), stellar interiors and stellar outflows, etc. Reynolds numbers of astrophysical flows are, typically, very high, which makes turbulence ubiquitous. Initially unmagnetized well-conductive turbulent fluid generates its own magnetic field which is dynamically important on almost all scales. In spiral galaxies large-scale dynamo generates ordered fields on the scale of the galactic disc, while in more symmetric objects, such as galaxy clusters, small-scale dynamo operates and generates fields on scales up to 20 kpc.

Inertial range of turbulence was introduced by Kolmogorov (1941) as a range of spatial scales where driving and dissipation are unimportant and perturbations exist due to energy transfer from one scale to another. In the inertial range of MHD turbulence perturbations of both velocity and magnetic field will be much smaller than the local Alfvénic velocity  $v_A = B/\sqrt{4\pi\rho}$ , due to the turbulence spectrum being steeper than  $k^{-1}$ , therefore local mean magnetic field will strongly affect dynamics in this range (Iroshnikov 1964; Kraichnan 1965). As in the case of hydrodynamics, the

study of MHD turbulence began with weakly compressible and incompressible cases which are directly applicable to many environments, such as stellar interiors, ICM and hot phases of the ISM. Later it was realized that many features of incompressible MHD turbulence are still present even in supersonic dynamics, due to the dominant effect of Alfvénic shearing (Cho & Lazarian 2003; Beresnyak et al. 2005). It had been pointed out by Goldreich & Sridhar (1995) that strong mean field incompressible turbulence is split into the cascade of Alfvénic mode, described by Reduced MHD or RMHD (Kadomtsev & Pogutse 1974; Strauss 1976) and the passive cascade of slow (pseudo-Alfvén) mode. In the strong mean field case it was sufficient to study only the Alfvénic dynamics, as it will determine all statistical properties of turbulence, such as spectrum or anisotropy. This decoupling was also observed in numerics. Luckily, being the limit of very strong mean field, RMHD has a precise two-parametric symmetry similar to the one in incompressible hydrodynamics (see §2), which, under certain conditions, makes universal cascade with power-law energy spectrum possible.

Interaction of Alfvénic perturbations propagating in a strong mean field is unusual due to a peculiar dispersion relation of Alfvénic mode,  $\omega = k_{\parallel}v_A$ , where  $k_{\parallel}$  is a wavevector parallel to the mean magnetic field. This results in a tendency of MHD turbulence to create “perpendicular cascade”, where the flux of energy is preferentially directed perpendicular to the magnetic field. This tendency enhances the nonlinearity of the interaction, described by  $\xi = \delta v k_{\perp}/v_A k_{\parallel}$ , which is the ratio of the mean-field term to the nonlinear term, and results in development of essentially strong turbulence. As turbulence becomes marginally strong,  $\xi \sim 1$ , the cascading timescales become close to the dynamical timescales  $\tau_{\text{casc}} \sim \tau_{\text{dyn}} = 1/\omega k_{\perp}$  and the perturbation frequency  $\omega$  has a lower bound due to an uncertainty relation  $\tau_{\text{casc}}\omega > 1$  (Goldreich & Sridhar 1995). This

<sup>1</sup> This rule have some notable exceptions, such as molecular gas in the early Universe, which could be fairly viscous on scales of protogalaxies or the solar wind which is not collisional enough to fully support compressible modes.

makes turbulence being “stuck” in the  $\xi \sim 1$  regime, which is known as “critical balance”. Note, that the above argument is not the only lower bound on  $\omega$ , with another bound being due to the *directional* uncertainty of the  $\mathbf{v}_A$ , which was discovered in Beresnyak & Lazarian (2008). In this paper we will mostly consider so-called “balanced” turbulence in which both uncertainties coincide. We refer the reader to Beresnyak & Lazarian (2008, 2009b) and references therein for the more general imbalanced case.

Goldreich-Sridhar model is predicting a  $k^{-5/3}$  energy spectrum with anisotropy described as  $k_{\parallel} \sim k_{\perp}^{2/3}$ . Numerical studies Cho & Vishniac (2000); Maron & Goldreich (2001) confirmed steep spectrum and scale-dependent anisotropy, but Maron & Goldreich (2001); Müller & Grappin (2005) claimed a shallower than  $-5/3$  spectral slope in the strong mean field case, which was close to  $-3/2$ . This motivated adjustments to the Goldreich-Sridhar model (Galtier et al. 2005; Boldyrev 2005; Gogoberidze 2007). A model with so called “dynamic alignment” (Boldyrev 2005, 2006) became popular after the scale-dependent alignment was discovered in numerical simulations (Beresnyak & Lazarian 2006). This model is based on the idea that the alignment between velocity and magnetic perturbations decreases the strength of the interaction scale-dependently, relying on the alignment being a power-law function of scale. This would, as they argue, modify the spectral slope of MHD turbulence from the  $-5/3$  Kolmogorov slope to the observed  $-3/2$  slope. It was also claimed that there is a self-consistent turbulent mechanism that produces such an alignment. In this paper we examine both the alignment and the spectrum. It turns out that the alignment is not universal and is tied to the driving scale. Also, spectral resolution study at high numerical resolutions favor  $-5/3$  spectral slope more than  $-3/2$  slope. It seems that earlier measurements of the MHD slope were premature and were not conducted with enough rigor.

In this paper we a) support our earlier claim of  $-5/3$  scaling and the measurement of Kolmogorov constant of Beresnyak (2011) and b) perform the measurement of anisotropy constant. These two measurements allow us to predict the local properties of MHD turbulence on any scale  $l$ , given only the dissipation rate  $\epsilon$  and Alfvén velocity  $v_A$ . Such a straightforward prediction is of great practical use for astrophysics.

## 2 BASIC EQUATIONS

Ideal MHD equations describe the dynamics of ideally conducting inviscid fluid with magnetic field and can be written in Heaviside and  $c = 1$  units as

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \rho(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} &= -\nabla P + \mathbf{j} \times \mathbf{B}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \partial_t \mathbf{B} &= \nabla \times (\mathbf{v} \times \mathbf{B}),\end{aligned}$$

with current  $\mathbf{j} = \nabla \times \mathbf{B}$  and vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ . This should be supplanted with energy equation and a prescription for pressure  $P$ . The incompressible limit assumes that the pressure is so high that the density is constant and velocity is purely solenoidal ( $\nabla \cdot \mathbf{v} = 0$ ). This does not necessarily

refer to the ratio of outer scale kinetic pressure to molecular pressure, but could be interpreted as scale-dependent condition. Indeed, if we go to the frame of the fluid, local perturbations of velocity will diminish with scale and will be much smaller than the speed of sound. In this situation it will be possible to decompose velocity into low-amplitude sonic waves and essentially incompressible component of  $\mathbf{v}$ , as long as we are not in the vicinity of a shock. The incompressible component, bound by  $\nabla \cdot \mathbf{v} = 0$ , will be described by much simpler equations:

$$\begin{aligned}\partial_t \mathbf{v} &= \hat{S}(-\boldsymbol{\omega} \times \mathbf{v} + \mathbf{j} \times \mathbf{b}), \\ \partial_t \mathbf{b} &= \nabla \times (\mathbf{v} \times \mathbf{b}),\end{aligned}$$

where we renormalized magnetic field to velocity units  $\mathbf{b} = \mathbf{B}/\rho^{1/2}$  (the absence of  $4\pi$  is due to Heaviside units) and used solenoidal projection operator  $\hat{S} = (1 - \nabla \Delta^{-1} \nabla)$  to get rid of pressure. Finally, in terms of Elsässer variables  $\mathbf{w}^{\pm} = \mathbf{v} \pm \mathbf{b}$  this could be rewritten as

$$\partial_t \mathbf{w}^{\pm} + \hat{S}(\mathbf{w}^{\mp} \cdot \nabla) \mathbf{w}^{\pm} = 0. \quad (1)$$

This equation resembles incompressible Euler’s equation. Indeed, hydrodynamics is just a limit of  $b = 0$  in which  $\mathbf{w}^+ = \mathbf{w}^-$ . This resemblance, however, is misleading, as the local mean magnetic field could not be excluded by the choice of reference frame and, as we noted earlier, will strongly affect dynamics on all scales. We can explicitly introduce local mean field as  $\mathbf{v}_A$ , assuming that it is constant, so that  $\delta \mathbf{w}^{\pm} = \mathbf{w} \pm \mathbf{v}_A$ :

$$\partial_t \delta \mathbf{w}^{\pm} \mp (\mathbf{v}_A \cdot \nabla) \delta \mathbf{w}^{\pm} + \hat{S}(\delta \mathbf{w}^{\mp} \cdot \nabla) \delta \mathbf{w}^{\pm} = 0. \quad (2)$$

In the linear regime of small  $\delta w$ ’s they represent perturbations, propagating along and against the direction of the magnetic field, with nonlinear term describing their interaction. As we noted earlier, due to the resonance condition of Alfvénic perturbations they tend to create more perpendicular structure, making MHD turbulence progressively more anisotropic. This was empirically known from tokamak experiments and was used in so-called reduced MHD approximation, which neglected parallel gradients in the nonlinear term (Kadomtsev & Pogutse 1974; Strauss 1976). Indeed, if we denote  $\parallel$  and  $\perp$  as directions parallel and perpendicular to  $\mathbf{v}_A$ , the mean field term  $(v_A \nabla_{\parallel}) \delta w^{\pm}$  is much larger than  $(\delta w_{\parallel}^{\mp} \nabla_{\parallel}) \delta w^{\pm}$  and the latter could be ignored in the inertial range where  $\delta w^{\pm} \ll v_A$ . This will result in Equation 2 being split into

$$\partial_t \delta \mathbf{w}_{\parallel}^{\pm} \mp (\mathbf{v}_A \cdot \nabla_{\parallel}) \delta \mathbf{w}_{\parallel}^{\pm} + \hat{S}(\delta \mathbf{w}_{\perp}^{\mp} \cdot \nabla_{\perp}) \delta \mathbf{w}_{\parallel}^{\pm} = 0, \quad (3)$$

$$\partial_t \delta \mathbf{w}_{\perp}^{\pm} \mp (\mathbf{v}_A \cdot \nabla_{\parallel}) \delta \mathbf{w}_{\perp}^{\pm} + \hat{S}(\delta \mathbf{w}_{\perp}^{\mp} \cdot \nabla_{\perp}) \delta \mathbf{w}_{\perp}^{\pm} = 0, \quad (4)$$

which, physically represent a limit of very strong mean field where  $\delta \mathbf{w}_{\parallel}^{\pm}$  is a slow (pseudo-Alfvén) mode and  $\delta \mathbf{w}_{\perp}^{\pm}$  is the Alfvén mode and Equation 3 describes a passive dynamics of slow mode which is sheared by the Alfvén mode, while Equation 4 describes essentially nonlinear dynamics of the Alfvén mode and is known as reduced MHD. For our purposes, to figure out asymptotic behavior in the inertial range, it is sufficient to study Alfvénic dynamics and slow

mode can be always added later, because it will have the same statistics.

It turns out that reduced MHD is often applicable beyond incompressible MHD limit, in a highly collisionless environments, such as tokamaks or the solar wind. This is due to the fact that Alfvén mode is transverse and does not require pressure support. Indeed, Alfvénic perturbations rely on magnetic tension as a restoring force and it is sufficient that charged particles be tied to magnetic field lines to provide inertia (Schekochihin et al. 2009).

A remarkable property of RMHD is that it has a precise two-parametric symmetry with respect to the anisotropy and the strength of the mean field:  $\mathbf{w} \rightarrow \mathbf{w}A$ ,  $\lambda \rightarrow \lambda B$ ,  $t \rightarrow tB/A$ ,  $\Lambda \rightarrow \Lambda B/A$ , which is similar to hydrodynamic symmetry. Here  $\lambda$  is a perpendicular scale,  $\Lambda$  is a parallel scale,  $A$  and  $B$  are arbitrary parameters of the transformation. It is due to this precise symmetry and the absence of any designated scale, that we can hypothesize universal regime, similar to hydrodynamic cascade of Kolmogorov (1941). In nature, the universal regime for MHD can be achieved with  $\delta w^\pm \ll v_A$ . In numerical simulations, we can directly solve RMHD equations, which have precise symmetry already built in. From practical viewpoint, the statistics from the full MHD simulation with  $\delta w^\pm \sim 0.1v_A$  is virtually indistinguishable from RMHD statistics and even  $\delta w^\pm \sim v_A$  is still fairly similar to the strong mean field case (Beresnyak & Lazarian 2009b).

### 3 BASIC SCALINGS

As was shown in a rigorous perturbation study of weak MHD turbulence, it has a tendency of becoming *stronger* on smaller scales (Galtier et al. 2000). Indeed, if  $k_\parallel$  is constant and  $k_\perp$  is increasing,  $\xi = \delta w k_\perp / v_A k_\parallel$  will increase, due to  $\delta w \sim k_\perp^{-1/2}$  in this regime. This will naturally lead to strong turbulence, where  $\xi$  will stuck around unity due to two competing processes: 1) increasing interaction by perpendicular cascade and 2) decrease of interaction due to the uncertainty relation  $\tau_{\text{casc}}\omega > 1$ , where  $\tau_{\text{casc}}$  is a cascading timescale. Therefore, MHD turbulence will be always marginally strong in the inertial range, which means that cascading timescale is associated with dynamical timescale  $\tau_{\text{casc}} \sim \tau_{\text{dyn}} = 1/\delta w k_\perp$  (Goldreich & Sridhar 1995). In this case, assuming that energy transfer is local in scale and, therefore, depend only on perturbations amplitude on each scale, we can write Kolmogorov-type phenomenology as

$$\epsilon^+ = \frac{(\delta w_\lambda^+)^2 \delta w_\lambda^-}{\lambda}, \quad \epsilon^- = \frac{(\delta w_\lambda^-)^2 \delta w_\lambda^+}{\lambda}, \quad (5)$$

where  $\epsilon^\pm$  is an energy flux of each of the Elsässer variables and  $\delta w_\lambda^\pm$  is a characteristic perturbation amplitude on a scale  $\lambda$ . Such an amplitude can be obtained by Fourier filtering with a dyadic filter in k-space, see, e.g., Beresnyak (2012).

Since we consider so-called balanced case with both  $w^\pm$  having the same statistical properties and energy fluxes, one of these equations is sufficient. This will result in a  $\delta w \sim \lambda^{1/3}$ , where  $\lambda$  is a perpendicular scale, or, in terms of energy spectrum  $E(k)$ ,

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}, \quad (6)$$

where  $C_K$  is known as Kolmogorov constant. We will be interested in Kolmogorov constant for MHD turbulence. This scaling is supposed to work until dissipation effects kick in. In our further numerical argumentation dissipation scale will play a big role, but not from a physical, but rather from a formal point of view. We will introduce an idealized scalar dissipation term in a RHS of Eq. 1 as  $-\nu_n (-\nabla^2)^{n/2} \mathbf{w}^\pm$ , where  $n$  is an order of viscosity and  $n = 2$  correspond to normal Newtonian viscosity, while for  $n > 2$  it is called hyperviscosity. The dissipation scale for this Goldreich-Sridhar model is the same as the one for Kolmogorov model, i.e.  $\eta = (\nu_n^3/\epsilon)^{1/(3n-2)}$ . This is a unique combination of  $\nu_n$  and  $\epsilon$  that has units of length. Note that Reynolds number, estimated as  $vL/\nu_2$ , where  $L$  is an outer scale of turbulence, is around  $(L/\eta)^{4/3}$ .

Furthermore, the perturbations of  $w$  will be strongly anisotropic and this anisotropy can be calculated from the critical balance condition  $\xi \approx 1$ , so that  $k_\parallel \sim k_\perp^{2/3}$ . Interestingly enough this could be obtained directly from units and the symmetry of RMHD equations from above. Indeed, in the RMHD limit,  $k_\parallel$  or  $1/\Lambda$  must be in a product with  $v_A$ , since only the product enters the original RMHD equations. We already assumed above that turbulence is local and each scale of turbulence has no knowledge of other scales, but only the local dissipation rate  $\epsilon$ . In this case the only dimensionally correct combination for the parallel scale  $\Lambda$ , corresponding to perpendicular scale  $\lambda$  is

$$\Lambda = C_A v_A \lambda^{2/3} \epsilon^{-1/3}, \quad (7)$$

where we introduced a dimensionless “anisotropy constant”  $C_A$ . Equations 6 and 7 roughly describe the spectrum and anisotropy of MHD turbulence. Note, that Goldreich-Sridhar’s  $-5/3$  is a basic scaling that should be corrected for intermittency. This correction is negative due to structure function power-law exponents being a concave function of their order (see, e.g., Frisch 1995) and is expected to be small in three-dimensional case. This correction for hydrodynamic turbulence is around  $-0.03$ . Such a small deviation should be irrelevant in the context of debate between  $-5/3$  and  $-3/2$ , which differ by about 0.17.

An alternative model was proposed by Boldyrev (2005, 2006) who suggested that these scalings are modified by a scale dependent factor that decreases the strength of the interaction, so that RHS of the Equation 5 is effectively multiplied by a factor of  $(l/L)^{1/4}$ , where  $L$  is an outer scale. We will discuss this hypothesis further in §8. In this case the spectrum will be expressed as  $E(k) = C_{K2} \epsilon^{2/3} k^{-3/2} L^{1/6}$ . Note that this spectrum is the only dimensionally correct spectrum with  $k^{-3/2}$  scaling, which does not contain dissipation scale  $\eta$ . The absence of  $L/\eta$ , is due to so-called zeroth law of turbulence which states that the amplitude at the outer scale should not depend on the viscosity. This law follows from the locality of energy transfer has been know empirically to hold very well. The dissipation scale of Boldyrev model is different from that of the Goldreich-Sridhar model and can be expressed as  $\eta' = (\nu_n^3/\epsilon)^{1/(3n-1.5)} L^{0.5/(3n-1.5)}$ .

**Table 1.** Three-dimensional RMHD simulations

Run	$n_x \cdot n_y \cdot n_z$	Dissipation	$\langle \epsilon \rangle$	$L/\eta$
R1	$256 \cdot 768^2$	$-6.82 \cdot 10^{-14} k^6$	0.073	200
R2	$512 \cdot 1536^2$	$-1.51 \cdot 10^{-15} k^6$	0.073	400
R3	$1024 \cdot 3072^2$	$-3.33 \cdot 10^{-17} k^6$	0.073	800
R4	$768^3$	$-6.82 \cdot 10^{-14} k^6$	0.073	200
R5	$1536^3$	$-1.51 \cdot 10^{-15} k^6$	0.073	400
R6	$384 \cdot 1024^2$	$-1.70 \cdot 10^{-4} k^2$	0.081	280
R7	$768 \cdot 2048^2$	$-6.73 \cdot 10^{-5} k^2$	0.081	560
R8	$768^3$	$-1.26 \cdot 10^{-4} k^2$	0.073	350
R9	$1536^3$	$-5.00 \cdot 10^{-5} k^2$	0.073	700

#### 4 THE SCALING ARGUMENT

Turbulence with very long ranges of scales is common in astrophysics. Three-dimensional numerics, however, are unable to reproduce such ranges and normally struggles to obtain even a small inertial range. In this situation we have to use rigorous quantitative arguments in order to investigate asymptotic scalings.

Suppose we performed several simulations with different Reynolds numbers. If we believe that turbulence is universal and the scale separation between forcing scale and dissipation scale is large enough, the properties of small scales should not depend on how turbulence was driven. This is because neither MHD nor hydrodynamic equations explicitly contain any scale, so simulation with a smaller dissipation scale could be considered, due to symmetries from §2, as a simulation with the same dissipation scale, but larger driving scale. E.g., the small scale statistics in a  $1024^3$  simulation should look the same as small-scale statistics in  $512^3$ , if the physical size of the elementary cell is the same and the dissipation scale is the same.

This scaling argument requires that the geometry of the elementary cells is the same and the actual numerical scheme used to solve the equations is the same. Also, numerical equations should not contain any scale explicitly, which is usually satisfied. The scaling argument does not require high precision on the dissipation scale or a particular form of dissipation, either explicit or numerical. This is because we only need the small-scale statistics to be similar in both simulations. This is achieved, on one hand, by applying the same numerical procedure, and, on the other hand, by turbulence locality that ensures that outer scale influence is small.

In practice, the scaling argument also known as a resolution study is done in the following way: the averaged spectra in two simulations are expressed in dimensionless units corresponding to the expected scaling, for example a  $E(k)k^{5/3}\epsilon^{-2/3}$  is used for hydrodynamics, and plotted versus dimensionless wavenumber  $k\eta$ , where dissipation scale  $\eta$  correspond to the same model, e.g.  $\eta = (\nu^3/\epsilon)^{1/4}$  is used for scalar second order viscosity  $\nu$  and Kolmogorov phenomenology. As long as the spectra are plotted this way and the scaling is correct, the curves obtained in simulations with different resolutions has to converge on small scales. Not only the spectrum, but any other statistical property of turbulence can be treated this way. This method has been used in hydrodynamics since long time ago, e.g. by Yeung & Zhou (1997); Gotoh et al. (2002); Kaneda et al. (2003). For hydrodynamics good convergence on the dissipation scale has

been obtained with rather moderate resolutions, which implies that hydrodynamic cascade has fairly narrow locality. When the resolution is increasing, the convergence is supposed to become better. This is because of the increased separation of scale between driving and dissipation. Better convergence means higher precision in discriminating different scalings, which was demonstrated in Kaneda et al. (2003), which directly measured the intermittency correction to the spectral slope. The optimum strategy for our MHD study is to perform the largest resolution simulations possible and do the scaling study as described in this section.

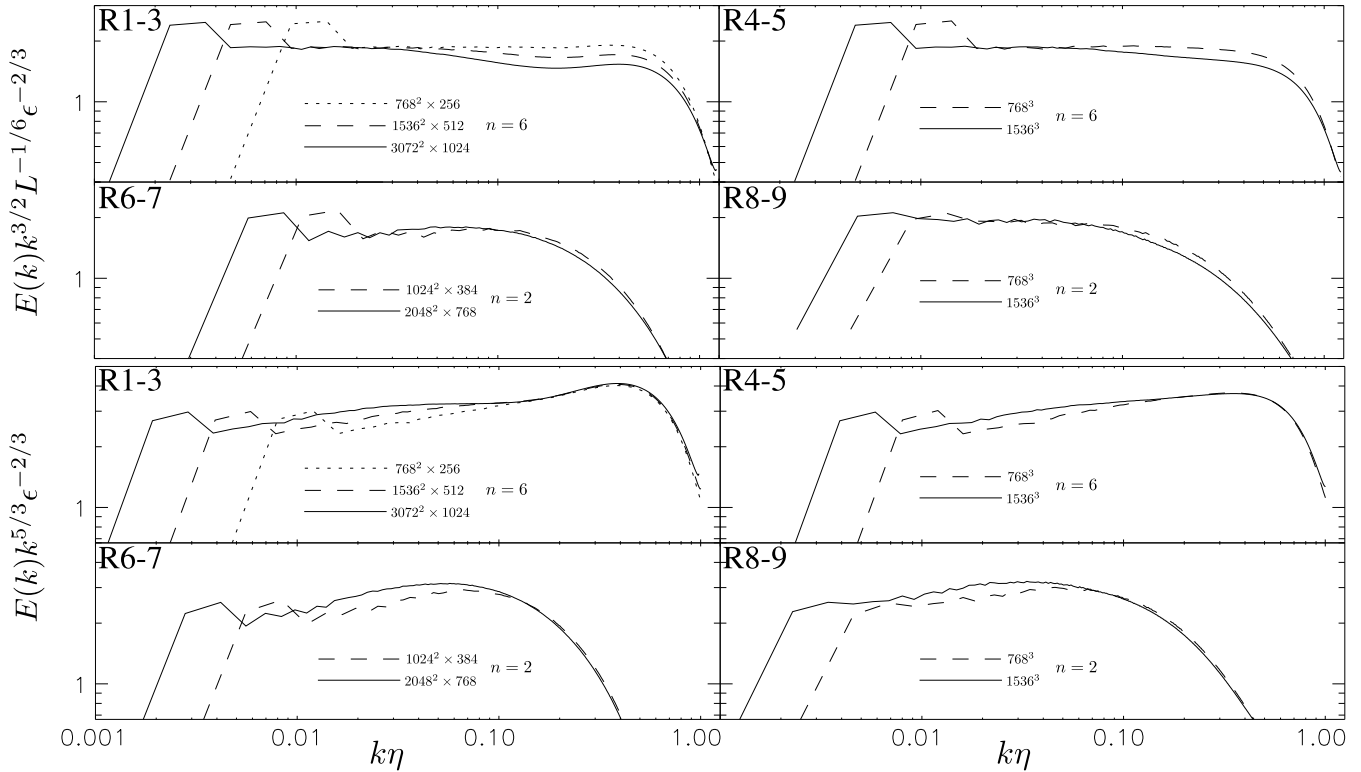
#### 5 NUMERICAL METHODS

We used pseudospectral dealiased code to solve RMHD equations. Same code was used earlier for RMHD, incompressible MHD and incompressible hydrodynamic simulations. The RHS of Eq. 4 was complemented by an explicit dissipation term  $-\nu_n(-\nabla^2)^{n/2}\mathbf{w}^\pm$  and forcing term  $\mathbf{f}$ . The code and the choice for numerical resolution, driving, etc, was described in great detail in our earlier publications (Beresnyak & Lazarian 2009b,a, 2010; Beresnyak 2011). Table 1 shows the parameters of the simulations. The Kolmogorov scale is defined as  $\eta = (\nu_n^3/\epsilon)^{1/(3n-2)}$ , the integral scale  $L = 3\pi/4E \int_0^\infty k^{-1}E(k)dk$  (which was approximately 0.79 for R1-3). Dimensionless ratio  $L/\eta$  could serve as a “length of the spectrum”, although spectrum is actually significantly shorter for  $n=2$  viscosity and somewhat shorter for  $n=6$  viscosity.

Since we would like to use this paper to illustrate the resolution study argument we used a variety of resolution, dissipation and driving schemes. There are four schemes, presented in Table 1, and used in simulations R1-3, R4-5, R6-7 and R8-9. In some of the simulations the resolution in the direction parallel to the mean magnetic field,  $n_x$ , was reduced by a factor compared to perpendicular resolution. This was deemed possible due to an empirically known lack of energy in the parallel direction in  $k$ -space and has been used before (see, e.g., Müller & Grappin 2005). The R4-5 and R8-9 groups of simulations were fully resolved in parallel direction. One would expect that roughly the same resolution will be required in parallel and perpendicular direction (Beresnyak & Lazarian 2009b). In all simulation groups time step was strictly inversely proportional to the resolution, so that we can utilize the scaling argument.

Driving had a constant energy injection rate for all simulations except R6-7, which had fully stochastic driving. All simulations except R8-9 had Elsässer driving, while R8-9 had velocity driving. All simulations were well-resolved and R6-7 were overresolved by a factor of 1.6 in scale (a factor of 2 in Re). The anisotropy of driving was that of a box, while injection rate was chosen so that the amplitude was around unity on outer scale, this roughly corresponds to critical balance on outer scale. Indeed, as we will show in subsequent section, since anisotropy constant is smaller than unity, our driving with  $\lambda \sim \Lambda \sim 1$  and  $\delta w \sim 1$  on outer scale is somewhat over-critical, so  $\Lambda$  decreases after driving scale to satisfy uncertainty relation (see Fig. 5). This is good for maintaining critical balance over wide range of scales as it eliminates possibility for weak turbulence.

In presenting four groups of simulations, with different



**Figure 1.** Numerical convergence of spectra in all simulations. Two upper rows are used to study convergence assuming Boldyrev model and two bottom rows – assuming Goldreich-Sridhar model. Note that definition of dissipation scale  $\eta$  depends on the model, see §3, this difference is tiny in hyperviscous simulations R1-5, but significant in viscous simulations R6-9. Numerical convergence requires that spectra will be similar on small scales, including the dissipation scale, see, e.g. Gotoh et al. (2002). As we see from the plots, numerical convergence is absent for Boldyrev model. For Goldreich-Sridhar model the convergence is reached only at the dissipation scale. Higher-resolution simulations are required to demonstrate convergence in the inertial range.

geometries of elementary cell, different dissipation terms and different driving, our intention is to show that the scaling argument works irrespective of numerical effects, but rather relies on scale separation and the assumption of universal scaling. Simulations R1-3 are the same as those presented in Beresnyak (2011). Largest simulation R3 with  $3.1 \times 10^9$  points used about 0.85 million CPU hours on TACC Ranger.

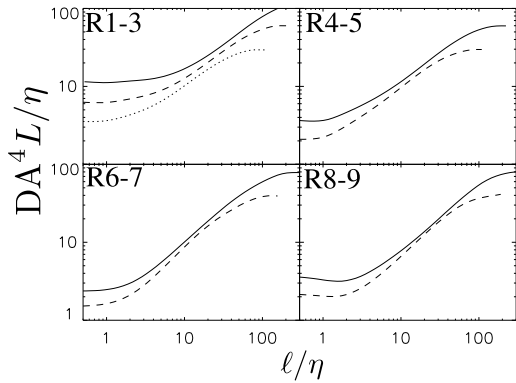
## 6 DRIVING AND CONSERVATION LAWS

All of our simulations except R8-9 used Elsässer driving. The choice of this driving is due to the formulation of our problem – we want to obtain asymptotic scalings in the inertial range of MHD turbulence. This is also the reason we use RMHD, although sometimes full MHD with strong mean field is used (Müller & Grappin 2005; Beresnyak & Lazarian 2009a,b). In this situation we simulate a tiny, compared to the outer scale, volume of turbulence. Since incompressible MHD has two Elsässer energy cascades and exchange of energy between these cascades is impossible, it makes sense that we provide energy to these two cascades independently, in fact this is the only way to simulate imbalanced turbulence in a periodic box (Beresnyak & Lazarian 2009b).

When driving is concerned, often the issue of integral conservation laws is raised. In particular, inviscid hydrodynamics has Kelvin’s circulation theorem which preserves circulation along any closed curve, moving with the fluid,

as long as the external force is potential. This also implies that the lines of vorticity are “frozen” into the fluid. Naturally, driving force in most hydrodynamic simulations is not potential and, as a result, Kelvin’s theorem is broken. Although non-potential forces are known to drive turbulence in nature, this usually is only important on the outer scale of turbulence, but in the inertial range such forces should be negligible compared to the dynamic pressure.

What is the physical meaning of volumetric force that is typically used in simulations which try to reproduce inertial range, such as Gotoh et al. (2002)? The key to understand this is to remember that each fluid element is embedded into the surrounding fluid that exerts forces on its boundaries. These forces supply energy that drives turbulence within this element. Similarly, in MHD fluid an external electromotive force (EMF) exists on the boundaries of the fluid element and the curl of this EMF enters in RHS of the induction equation. Unfortunately, simulating fluid element embedded into a larger system would effectively mean simulating this larger system and is not practical. In practice we use periodic boxes to mimic statistical homogeneity. Despite driving in such boxes being restricted to low wavenumbers, the correlation scale is typically smaller than the box size by a factor of 3-4, which simulates many fluid elements in a box. The driving around  $k=2.3$ , therefore, emulates an action exerted on the boundaries of these fluid elements. If  $k_{\max}$  is a maximum wavenumber of driving, then the change



**Figure 2.** Resolution study for “dynamic alignment”, assuming Boldyrev scaling. Both axis are dimensionless, solid is higher resolution and dashed is lower resolution. Convergence is absent for all simulations. This suggests that  $l^{0.25}$  is not a universal scaling for alignment.

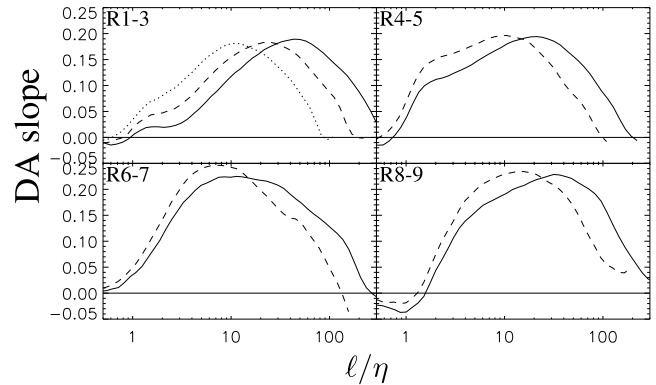
of circulation within a loop of size  $l < 2\pi/k_{\max}$ , associated with forcing, is going to be small, emulating the situation of real turbulence where the forces exerted on the boundaries of fluid element will actually preserve circulation along any loop within this element.

Similarly, Alfvén’s theorem is a conservation of the magnetic flux through a given loop and this implies that the magnetic field lines are frozen into the fluid. It can be broken by an external EMF, however, if such an EMF is restricted to low wavenumbers in a simulation with periodic box, this will effectively emulate an external EMF action on the boundaries of the fluid element. There is a full analogy between Alfvén’s theorem and Kelvin’s theorem since we expect neither significant external forces nor external EMF at the inertial range scales. Also, it turns out that there is an analogy between the breakdown of Kelvin’s theorem and Alfvén’s theorem in turbulent flows (Eyink & Aluie 2006; Eyink 2007).

To summarize, the driving for a simulation with a strong mean field or RMHD simulation reproducing inertial range, must have Elsässer driving, i.e. independent driving of  $w^+$  and  $w^-$ . This will simulate supply of Elsässer energies from larger scales. If turbulence is strong, a pure velocity driving is also possible due to the quick nonlinear decorrelation of  $w^+$  and  $w^-$  eddies. Volumetric Elsässer driving does not honor Kelvin’s or Alfvén’s theorems, however, by restricting driving to large scales we effectively emulate the action of external forces which conserve fluxes and circulations. On the dissipation scales Kelvin’s or Alfvén’s theorems are broken by viscosity and magnetic diffusivity and in the inertial range they will be broken by turbulence. Therefore, there is an analogy between forced viscous hydrodynamic simulations in a periodic box and forced dissipative MHD simulations.

## 7 SPECTRA

Fig. 1 presents a resolution study all simulations. The upper rows assume Boldyrev scaling, while the bottom rows assume Goldreich-Sridhar scaling. Reasonable convergence on small scales was achieved only for Goldreich-Sridhar scaling.



**Figure 3.** DA slope, defined as  $l/DA\partial DA/\partial l$ , solid is higher resolution and dashed is lower resolution. Dynamic alignment slope does not converge and has a tendency of becoming smaller in higher-resolution simulations. This may indicate that the asymptotic alignment slope is zero, which will correspond to the GS95 model.

The normalized amplitude at the dissipation scale for two upper rows of plots systematically goes down with resolution, suggesting that  $-3/2$  is not an asymptotic scaling. The flat part of the normalized spectrum on R1-3 plots was fit to obtain Kolmogorov constant of  $C_{KA} = 3.27 \pm 0.07$  which was reported in Beresnyak (2011). The total Kolmogorov constant for both Alfvén and slow mode in the above paper was estimated as  $C_K = 4.2 \pm 0.2$  for the case of isotropically driven turbulence with zero mean field, where the energy ratio of slow and Alfvén mode  $C_s$  is between 1 and 1.3. This larger value  $C_K = C_{KA}(1 + C_s)^{1/3}$  is due to slow mode being passively advected and not contributing to nonlinearity. The measurement of  $C_{KA}$  had relied on an assumption that the region around  $k\eta \approx 0.07$  represent asymptotic regime. It is possible, though unlikely, that  $C_{KA}$  is slightly underestimated due to this region is somewhat lower than the asymptotic regime due to antibottleneck effect exacerbated by a reduced parallel resolution. The antibottleneck effect, however is typically much smaller than the bottleneck effect, so this correction is probably within the stated errorbars. Next simulation in a R4-5 series (3072<sup>3</sup>) should make this clear. As far as simulations with normal viscosity R6-9 go, it seems impossible to approach good inertial range until resolutions of at least 4096<sup>3</sup>.

## 8 DYNAMIC ALIGNMENT

Boldyrev (2005) suggested that  $\mathbf{w}^+$  and  $\mathbf{w}^-$  eddies are systematically aligned. This was investigated numerically in in Beresnyak & Lazarian (2006) and no significant alignment was found for the averaged angle between  $\mathbf{w}^+$  and  $\mathbf{w}^-$ ,  $AA = \langle |\delta \mathbf{w}_\lambda^+ \times \delta \mathbf{w}_\lambda^-| / (|\delta \mathbf{w}_\lambda^+| |\delta \mathbf{w}_\lambda^-|) \rangle$ , but when this angle was weighted with the amplitude  $PI = \langle |\delta \mathbf{w}_\lambda^+ \times \delta \mathbf{w}_\lambda^-| \rangle / (|\delta \mathbf{w}_\lambda^+| |\delta \mathbf{w}_\lambda^-|)$ , some alignment was found. Then Boldyrev (2006) suggested alignment between  $\mathbf{v}$  and  $\mathbf{b}$  and (Mason et al. 2006) suggested a particular amplitude-weighted measure,  $DA = \langle |\delta \mathbf{v}_\lambda \times \delta \mathbf{b}_\lambda| \rangle / (|\delta \mathbf{v}_\lambda| |\delta \mathbf{b}_\lambda|)$ . We note that DA is similar to PI but contain two effects: alignment and local imbalance. The latter could be mea-

sured with  $IM = \langle |\delta(w_\lambda^+)^2 - \delta(w_\lambda^-)^2| \rangle / \langle \delta(w_\lambda^+)^2 + \delta(w_\lambda^-)^2 \rangle$ , Beresnyak & Lazarian (2009a).

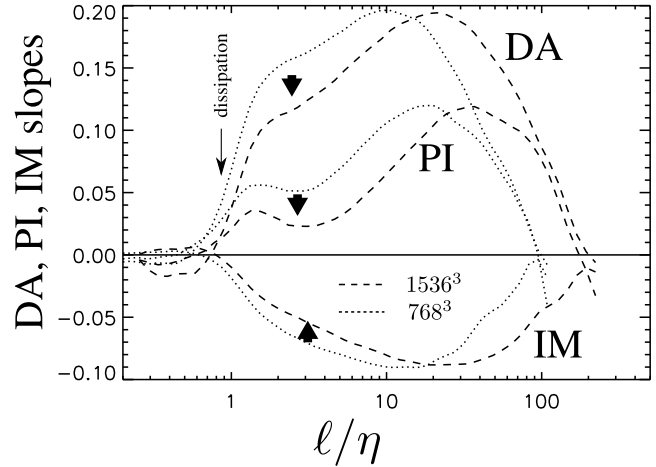
In this section we check the assertion of Boldyrev (2005, 2006) that alignment depends on scale as  $\lambda^{1/4}$ , by using DA which is, by some reason, favored by aforementioned group. We did a resolution study of DA, assuming suggested scaling, which is presented on Figure 2. Convergence was absent in all simulations. Note, that previous studies that claimed that there is a good correspondence with Boldyrev model did not perform the scaling study, therefore these claims are not well substantiated. A result from a single isolated simulation could be easily contaminated by the effects of outer scale, since it is not known a-priori how local MHD turbulence is and what resolution is sufficient to get rid of such effects. On the contrary, the resolution study offers a systematic approach to this problem.

Fig. 3 shows “dynamic alignment” slope for all simulations. Although there is no convergence as in the previous plot, it is interesting to note that alignment slope decreases with resolution. This suggests that most likely the asymptotic state for the alignment slope is zero, i.e. alignment is scale-independent and Goldreich-Sridhar model is recovered. Also, alignment from simulations R1-5 seems to indicate that the maximum of the alignment slope is tied to the outer scale, therefore alignment is a transitional effect.

In our earlier studies (Beresnyak & Lazarian 2006, 2009a) we measured several types of alignment and found no evidence that all alignment measures follow the same scaling, see, e.g., Fig. 4. As one alignment measure, PI, has been already known to be scale-dependent (Beresnyak & Lazarian 2006) prior to DA, it appears that a particular measure of the alignment in Mason et al. (2006) was been picked for being most scale-dependent and no thorough explanation was given why it was preferred. Furthermore, it was claimed that DA has the asymptotic scaling of  $\sim (l/L)^{1/4}$  and at the same time it is an interaction weakening factor that will result in a  $-3/2$  spectrum. This is unlikely, since the interaction weakening factor will appear from a third-order structure function, while DA is a correction factor based on a ratio of second-order structure functions. Intermittency corrections are supposed to be small, however. More importantly, a physical justification of DA and its preference over other measures, such as PI or IM, that differ from DA quite significantly, was lacking.

We are not aware of any convincing physical argumentation explaining why alignment should necessarily be a power-law of scale. Boldyrev (2006) argues that alignment will tend to increase, but will be bounded by field wandering, i.e. the alignment on each scale will be created independently of other scales and will be proportional to the relative perturbation amplitude  $\delta B/B$ . But this violates two-parametric symmetry of RMHD equations mentioned above, which suggests that field wandering can not destroy alignment or imbalance. Indeed, a perfectly aligned state, e.g., with  $\delta \mathbf{w}^- = 0$  is a precise solution of MHD equations and it is not destroyed by its own field wandering. The alignment measured in simulations of strong MHD turbulence with different values of  $\delta B_L/B_0$  showed very little or no dependence on this parameter (Beresnyak & Lazarian 2009a).

Why alignment measures are scale-dependent quantities over about one order of magnitude in scale is an interesting question. Most plausible explanation is that because



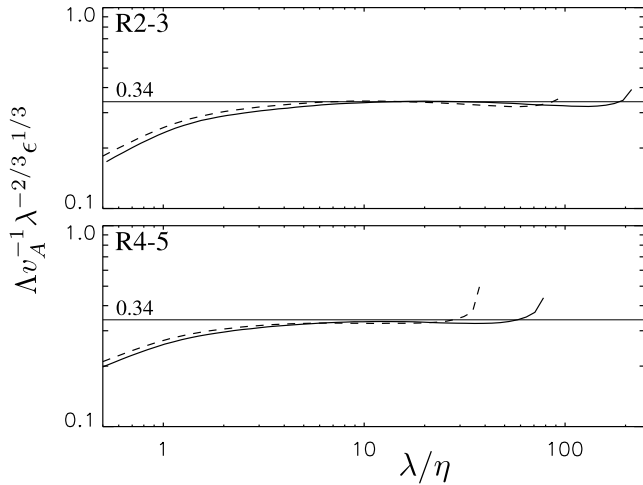
**Figure 4.** Slopes of several alignment measures vs scale in R4-5 (for definitions see the text). Each measure follows its own scaling, however there are indications that they are all tied to the outer scale, due to the maximum of alignment being a fraction of the outer scale, which is an indication that their scale-dependency is of transient nature.

MHD turbulence is much less local than hydro turbulence (Beresnyak & Lazarian 2009a, 2010; Beresnyak 2011) and because driving does not mimic the properties of the inertial range, the transition to asymptotic statistics is very wide and many quantities appear as scale-dependent, while they are simply adjusting to the asymptotic regime.

The contribution to energy flux from different  $k$  wavebands is important to understand, since most cascade models assume locality, or rather to say the very term “cascade” assumes locality. An analytical upper bound on locality suggests that the width of the energy transfer window can scale as  $C_K^{9/4}$  (Beresnyak 2012), however, in practice turbulence can be more local. The observation of Beresnyak & Lazarian (2009a) that MHD simulations normally lack bottleneck effect, even with high-order dissipation, while hydrodynamic simulations always have bottleneck, which is especially dramatic with high-order dissipation, is consistent with above conjecture on locality, since bottleneck effect relies on locality of energy transfer. As locality constraint depends on the efficiency of the energy transfer, so that the efficient energy transfer must be local, while inefficient one could be nonlocal (Beresnyak & Lazarian 2010; Beresnyak 2011, 2012). As we observe larger  $C_K$  in MHD turbulence compared to hydrodynamic turbulence, the former could be less local than the latter, which is consistent with our earlier findings.

## 9 ANISOTROPY

In Section 3 we suggested that anisotropy should be universal in the inertial range and expressed as  $\Lambda = C_{AVA} \lambda^{2/3} \epsilon^{-1/3}$ , where  $C_A$  is an anisotropy constant to be determined from the numerical experiment or observation. Note, that both Alfvénic and slow modes should have the same anisotropy. This is because they have the same ratio of propagation to nonlinear timescales. Figure 5 shows anisotropy for the two best resolved groups R1-3 and R4-5. We used a model independent method of



**Figure 5.** The scaling study for anisotropy shows moderately good convergence to a universal anisotropy  $\Lambda = C_A v_A \lambda^{2/3} \epsilon^{-1/3}$  with anisotropy constant  $C_A$  of around 0.34.

minimum parallel structure function, described in detail in Beresnyak & Lazarian (2009b). Alternative definitions of local mean field give comparable results, as long as they are reasonable. The convergence of anisotropy curves at the dissipation scale is not as good as convergence of spectra in R1-5 on the dissipation scale. This is due to this measure being calculated from structure functions which are less local in scale than 3D spectra, i.e. dissipation scale is still being somewhat affected by the transition to asymptotic regime. From R1-3 we obtain  $C_A = 0.34$ . Note, that the conventional definition of critical balance involve the amplitude, rather than  $(\epsilon\lambda)^{1/3}$ , so the constant in this classical formulation will be  $C_A C_K^{1/2} \approx 0.63$ , which is closer to unity. Together with energy spectrum this is a full description of universal axisymmetric two-dimensional spectrum of MHD turbulence in the inertial range.

## 10 SUMMARY AND DISCUSSION

In this paper we argued that the properties of Alfvén and slow components of MHD turbulence in the inertial range will be determined only by the Alfvén speed  $v_A$ , dissipation rate  $\epsilon$  and the scale of interest  $\lambda$ . The energy spectrum and anisotropy of Alfvén mode will be expressed as

$$E(k) = C_K \epsilon^{2/3} k^{-5/3},$$

$$\Lambda/\lambda = C_A v_A (\lambda \epsilon)^{-1/3},$$

with  $C_K = 3.3$  and  $C_A = 0.34$ . If the slow mode is present, its anisotropy will be the same, and it will contribute to both energy and dissipation rate. Assuming the ratio of slow to Alfvén energies between 1 and 1.3, the latter was observed in statistically isotropic high resolution MHD simulation with no mean field, we can use  $C_K = 4.2$  for the total energy spectrum (Beresnyak 2011).

Anisotropy of MHD turbulence is an important property that affects such processes as interaction with cosmic

rays, see, e.g., Yan & Lazarian (2002). Since cosmic ray pressure in our Galaxy is of the same order as dynamic pressure, their importance should not be underestimated. Another process affected is the three-dimensional turbulent reconnection, see, e.g., Lazarian & Vishniac (1999).

Previous measurements of the energy slope relied on the highest-resolution simulation and fitted the slope in the fixed  $k$ -range close to the driving scale, typically between  $k = 5$  and  $k = 20$ . We argue that such a fit is unphysical unless a numerical convergence has been demonstrated. We can plot the spectrum vs dimensionless  $k\eta$  and if we clearly see a converged dissipation range and a bottleneck range, we can assume that larger scales, in terms of  $k\eta$  represent inertial range. In fitting fixed  $k$ -range at low  $k$  we will never get rid of the influence of the driving scale. In fitting a fixed  $k\eta$  range, the effects of the driving will diminish with increasing resolution.

Since we still have trouble transitioning into the inertial range in large mean field simulations, for now it is impossible to demonstrate inertial range in statistically isotropic simulations similar to once presented in Müller & Grappin (2005). This is because we do not expect a universal power-law scaling in trans-Alfvénic regime, due to the absence of appropriate symmetries and the transitioning to sub-Alfvénic regime, where such scaling is possible, will require some extra scale separation. These two transitions require numerical resolution that is even higher than the highest resolution presented in this paper and for now seem computationally impossible.

Full compressible MHD equations contain extra degrees of freedom, which, in a weakly compressible case, entails the additional cascade of the fast MHD mode, possibly of weak nature. Supersonic simulations with moderate Mach numbers (Cho & Lazarian 2003) show that Alfvénic cascade is pretty resilient and is not much affected by compressible motions. The models of the “universal” supersonic turbulence covering supersonic large scales and effectively subsonic small scales are based mainly on simulations with limited resolution and unlikely to hold true. This is further reinforced by the results of this paper which demonstrated that even a much simpler case of sub-Alfvénic turbulence require fairly high resolutions to obtain an asymptotic scaling.

In this paper we treated so called balanced case with  $\delta w^+ \approx \delta w^-$ . A more general imbalanced case has been discussed in Beresnyak & Lazarian (2008, 2009b, 2010), see also references therein.

### 10.1 Results of Grappin & Müller (2010)

Grappin & Müller (2010) observed that MHD turbulence have scale-independent anisotropy, in contrast with the Goldreich-Sridhar model and our own results. We believe that this measurement is correct, in fact, similar 2D spectra has been reported in our study Beresnyak & Lazarian (2009b). However, these are 2D spectra obtained with respect to the global mean magnetic field. A trivial exercise (Cho et al. 2002; Beresnyak & Lazarian 2009a) show that measuring anisotropy with respect to the global field will destroy scale-dependent anisotropy, even in the case of very strong field. Indeed, even if we have  $\delta B_L/B_0 \ll 1$ , the anisotropy in the global frame will be limited from above by  $B_0/\delta B_L$ , due to field wandering, while the Goldreich-Sridhar



anisotropy on the scale  $l$  will be much higher,  $\sim B_0/\delta B_l$ , by a factor of  $B_L/B_l$ . In particular, each individual volume on scale  $l$  will have anisotropy with respect to the mean field averaged on the same volume, however, the direction of this field will deviate from the global mean field statistically, with RMS deviation of around  $\delta B_L/B_0$ . This directional deviations will easily destroy anisotropy  $\sim B_0/\delta B_l$  if we average spectra measured with respect to  $B_0$ . Therefore, one must measure anisotropy with respect to local mean field, which was realized in Cho & Vishniac (2000). From this argument, we see that the local mean field must be averaged on scales smaller or equal to  $l$ , a scale at which spectrum or structure function is measured and  $l$  itself is a preferred averaging scale. Also, it is anisotropy with respect to the local mean field, which is important for cosmic ray dynamics (Yan & Lazarian 2002, 2004).

## 10.2 Results of Mason et al. (2011)

Mason et al. (2011) conducted a series of low-resolution simulations ( $256^3 - 512^3$ ) studying the effect on dynamic alignment by numerics on viscous scale, e.g., by changing resolution with constant  $Re$ , or changing elementary cell geometry, such as the ratio of parallel to perpendicular resolution. Their conclusion was that alignment is, indeed, somewhat influenced by numerics.

As we explained previously, numerical effects on viscous scale are irrelevant for scaling study, if latter was conducted properly. Therefore it was quite puzzling that Mason et al. (2011) claimed that their results invalidate Beresnyak (2011), which did a proper scaling study with high resolution simulations. Furthermore, it is also puzzling that Mason et al. (2011) claimed an “extended scaling law” for alignment (previously mentioned  $\lambda^{1/4}$ ), despite the fact that they did not even attempt a proper scaling study for this quantity and did not demonstrate convergence.

Although studying obscure effects on viscous scales is pretty common in modern numerical studies of hydrodynamic turbulence, we will argue that such studies in MHD are of limited value for astrophysics, since dissipation mechanism of astrophysical plasma is, typically, not a scalar second-order diffusion.

On the contrary, an asymptotic scalings of the inertial range of MHD turbulence is of high value. However, such scalings could only be demonstrated by a proper rigorous scaling study in the spirit of Yeung & Zhou (1997); Gotoh et al. (2002), and not by arbitrarily assigning “inertial range” to a fixed range in wavenumbers.

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